Int J Theor Phys (2010) 49: 1446–1451
DOI 10.1007/s10773-010-0324-3

Obtaining Multimode Entangled State Representation by Generalized Radon Transformation of the Wigner Operator

Xue-Fen Xu

Received: 21 January 2010 / Accepted: 18 March 2010 / Published online: 8 April 2010

Abstract In a preceding paper (Fan and Lv in J. Math. Phys. 50:102108, 2009), the phase-space integration corresponding to the straight line characteristic of two different real parameters λ, τ over the Wigner operator (i.e. the Radon transformation) leads to pure-state density operator $|u\rangle_{\lambda,\tau\,\lambda,\tau}\langle u|$, where $|u\rangle_{\lambda,\tau}$ is just the coordinate-momentum intermediate representation. In this work we show that generalized Radon transformation of the Wigner operator yields multimode density operator of continuum variables. This provides us with a new approach for obtaining multimode entangled state representation. The Weyl ordering of the Wigner operator is used in our discussions.

Keywords Radon transform · Weyl correspondence · Multimode entangled state representation

In recent years tomographic methods of reconstructing the Wigner function based on the Radon transform [1, 2] has brought much attention of physicists [3–5]. Ellinas and Bracken, corresponding to the straight line $(q\cos\theta + p\sin\theta)$ in phase space, has introduced so-called the region operators [6]

$$\widehat{R}(u,\theta) = \frac{1}{\pi}\int \delta(u - q\cos\theta - p\sin\theta)\widehat{\Delta}(q,p)dqdp, \tag{1}$$

where $\widehat{\Delta}(q,p)$ is Wigner operator. In Ref. [7] Fan and Lv developed strategies for the construction of region operators so that they appear as pure state density operator and can be more conveniently applied in physics. They used the normal ordering form [8, 9]

$$\widehat{\Delta}(q,p) = \frac{1}{\pi} :e^{-(\hat{Q}-q)^2-(p-\widehat{P})^2}: \tag{2}$$

Project supported by the National Science Foundation of China (Grant No 10774108), and the Research Foundation of the Jiangsu Teachers University of Technology.

X.-F. Xu (✉)
School of Mathematics and Physics,
Jiangsu Teachers University of Technology, Changzhou, Jiangsu 213001, China
e-mail: xuxf@jstu.edu.cn

and the IWOP technique [10] to show

$$\begin{aligned}&\int dqdp\delta(u-\lambda q-\tau p)\widehat{\Delta}(q,p)\\&=[\pi(\lambda^2+\tau^2)]^{-1/2}\colon\ \exp\left[\frac{-1}{\lambda^2+\tau^2}(u-\lambda\hat{Q}-\tau\widehat{P})^2\right]\colon\\&=|u\rangle_{\lambda,\tau\,\lambda,\tau}\langle u|,\end{aligned}\tag{3}$$

where the new state $|u\rangle_{\lambda,\tau}$ is

$$|u\rangle_{\lambda,\tau}=[\pi(\lambda^2+\tau^2)]^{-1/4}\exp\left\{-\frac{u^2}{2(\lambda^2+\tau^2)}+\frac{\sqrt{2}\widehat{A}^{\dagger}u}{\lambda-i\tau}-\frac{\lambda+i\tau}{2(\lambda-i\tau)}\widehat{A}^{\dagger 2}\right\}|0\rangle,\tag{4}$$

which is just the eigenstate of $\lambda\widehat{Q}+\tau\widehat{P}$,

$$(\lambda\hat{Q}+\tau\widehat{P})|u\rangle_{\lambda,\tau}=u|u\rangle_{\lambda,\tau},\tag{5}$$

here

$$\hat{Q}=\frac{\widehat{A}+\widehat{A}^{\dagger}}{\sqrt{2}},\quad \hat{P}=\frac{\widehat{A}-\widehat{A}^{\dagger}}{\sqrt{2}i},\quad [\widehat{A},\widehat{A}^{\dagger}]=1.\tag{6}$$

$|u\rangle_{\lambda,\tau}$ is complete, since

$$\int du|u\rangle_{\lambda,\tau\,\lambda,\tau}\langle u|=\frac{1}{\sqrt{\pi(\lambda^2+\tau^2)}}\int du\colon\ \exp\left[-\frac{1}{\lambda^2+\tau^2}(u-\lambda\hat{Q}-\tau\widehat{P})^2\right]\colon\ =1.\tag{7}$$

Besides, it is also orthonormal

$$_{\lambda,\tau}\langle u'|u\rangle_{\lambda,\tau}=\delta(u-u'),\tag{8}$$

so $|u\rangle_{\lambda,\tau}$ is qualified to make up a new quantum mechanical representation, by observing (5) we name it the coordinate-momentum intermediate representation. The purpose of this work is to remark that the generalized Radon transformation of the Wigner operator yields multimode entangled state density operator of continuum variables. This provides us with a new approach for obtaining the multimode entangled state representation. The Weyl ordering of the Wigner operator is used in our discussions.

Theorem *Assuming that $|r\rangle$ is the eigenvector of an Hermite operator $\hat{R}\equiv\hat{R}(\hat{Q},\hat{P})$, $\hat{R}|r\rangle=r|r\rangle$, $\langle r|r'\rangle=\delta(r-r')$, and assuming that $\hat{R}$ is in Weyl ordering, which means $\hat{R}=\vdots\,\hat{R}\,\vdots$, ($\vdots\ \vdots$ denotes Weyl ordering symbol), then*

$$|r\rangle\langle r|=\int dpdq\delta[r-R(q,p)]\widehat{\Delta}(q,p),\tag{9}$$

where $R(q,p)$ is the classical Weyl function of $\hat{R}$ and $\widehat{\Delta}(q,p)$ is Wigner operator with q, p being classical coordinate and momentum.

Proof According to the Weyl correspondence rule [11]

$$\hat{R}=\int dpdqR(q,p)\widehat{\Delta}(q,p),\tag{10}$$

we have

$$|r\rangle = \int dr'\delta(r-r')|r'\rangle = \int dr'\delta(r-\hat{R})|r'\rangle$$
$$= \int dr' \vdots\, \delta(r-\hat{R}) \,\vdots\, |r'\rangle$$
$$= \int dr' \int dpdq\delta[r-R(q,p)] \vdots\, \delta(q-\hat{Q})\delta(p-\hat{P}) \,\vdots\, |r'\rangle, \tag{11}$$

where we have considered the condition $\vdots\, \hat{R} \,\vdots = \hat{R}$ and used the properties of the Dirac-detal function.

Due to the Weyl ordered form of the Wigner operator [12]

$$\widehat{\Delta}(q,p) = \vdots\, \delta(q-\hat{Q})\delta(p-\hat{P}) \,\vdots = \vdots\, \delta(p-\hat{P})\delta(q-\hat{Q}) \,\vdots, \tag{12}$$

it follows that

$$|r\rangle = \int dr' \int dpdq\delta[r-R(q,p)]\widehat{\Delta}(q,p)|r'\rangle. \tag{13}$$

Comparing $|r\rangle = \int dr'|r\rangle\langle r|r'\rangle$ yields (9). □

On the other hand, according to the Weyl rule and(12)

$$\int dpdq\delta[r-R(q,p)]\widehat{\Delta}(q,p) = \vdots\, \delta[r-\hat{R}(\hat{Q},\hat{P})] \,\vdots \tag{14}$$

so (9) is equal to

$$|r\rangle\langle r| = \vdots\, \delta[r-\hat{R}(\hat{Q},\hat{P})] \,\vdots. \tag{15}$$

Lemma *For the multimode case, assuming that $|\vec{r}\rangle \equiv |r_1, r_2, \ldots, r_n\rangle$ is the common eigenvector of n mutually commutable Hermite operators $\hat{R}_i(\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_n, \hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n)$, $\hat{R}_i|r_1, r_2, \ldots, r_n\rangle = r_i|\vec{r}\rangle$, $\langle\vec{r}|\vec{r}'\rangle = \delta(\vec{r}-\vec{r}')$, $i = 1, 2, \ldots, n$, and assuming that each $\hat{R}_i$ is in Weyl ordering, then following the same procedures as proving (9) we have*

$$|r_1, r_2, \ldots, r_n\rangle\langle r_1, r_2, \ldots, r_n| = \int \prod_{i=1}^{n} dp_i dq_i \delta[r_i - R_i(q_i, p_i)]\widehat{\Delta}_i(q_i, p_i), \tag{16}$$

where

$$\prod_{i=1}^{n} \widehat{\Delta}_i(q_i, p_i) = \frac{1}{\pi^n} \colon \exp\left[-\sum_j^n [(\hat{Q}_j - q_j)^2 + (p_j - \widehat{P}_j)^2]\right] \colon \equiv \widehat{\Delta}_n \tag{17}$$

is the n-mode Wigner operator.

Examples

Equations (9) and (16) involve Dirac-delta function, so they are generalized Radon transformation of the Wigner operator, which can provide us with a new approach for obtaining multimode entangled state representation, as we shall demonstrate in the following.

According to the theorem (16) and (17) we should have the following Radon transformation of the Wigner operator $\hat{\Delta}_2$,

$$|\xi\rangle\langle\xi| = \int \prod_{i=1}^{2} dq_i dp_i \delta\left[\xi_1 - \frac{(q_1+q_2)}{\sqrt{2}}\right]\delta\left[\xi_2 - \frac{(p_1-p_2)}{\sqrt{2}}\right]\hat{\Delta}_2, \tag{18}$$

where $|\xi\rangle$ with $\xi = \xi_1 + i\xi_2$ is the eigenvector of $\hat{Q}_1 + \hat{Q}_2$ and $\hat{P}_1 - \hat{P}_2$, since (q_1+q_2) and (p_1-p_2) are the Weyl correspondence of $\hat{Q}_1 + \hat{Q}_2$ and $\hat{P}_1 - \hat{P}_2$, respectively. To know the explicit form of $|\xi\rangle$, we use (18) and

$$\int \exp(-ax^2+bx)dx = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right), \quad a>0 \tag{19}$$

to calculate

$$\begin{aligned}
|\xi\rangle\langle\xi| &= \int dq_1dp_1dq_2dp_2\delta\left[\xi_1 - \frac{(q_1+q_2)}{\sqrt{2}}\right]\delta\left[\xi_2 - \frac{(p_1-p_2)}{\sqrt{2}}\right] \\
&\quad \times \frac{1}{\pi^2} : \exp[-(q_1-\hat{Q}_1)^2 - (p_1-\hat{P}_1)^2 - (q_2-\hat{Q}_2)^2 - (p_2-\hat{P}_2)^2] : \\
&= \frac{2}{\pi^2}\int dq_2dp_2 : \exp[-(\sqrt{2}\xi_1 - q_2 - \hat{Q}_1)^2 - (\sqrt{2}\xi_2 + p_2 - \hat{P}_1)^2 \\
&\quad - (q_2-\hat{Q}_2)^2 - (p_2-\hat{P}_2)^2] : \\
&= \frac{1}{\pi} : \exp\left[-\left(\xi_1 - \frac{\hat{Q}_1+\hat{Q}_2}{\sqrt{2}}\right)^2 - \left(\xi_2 - \frac{\hat{P}_1-\hat{P}_2}{\sqrt{2}}\right)^2\right] : .
\end{aligned} \tag{20}$$

According to

$$\hat{Q}_j = (a_j + a_j^\dagger)/\sqrt{2}, \qquad \hat{P}_j = (a_j - a_j^\dagger)/(i\sqrt{2}), \tag{21}$$

where $[a_j, a_k^\dagger] = \delta_{jk}$, and noticing the normal product form of vacuum projector

$$: \exp(-a_1^\dagger a_1 - a_2^\dagger a_2) : = |00\rangle\langle 00|, \tag{22}$$

we split the right hand side of (20) as the form $F(a_i^\dagger)|00\cdots 0\rangle\langle 00\cdots 0|F(a_i)$ and derive

$$|\xi\rangle = \frac{1}{\sqrt{\pi}}\exp[-|\xi|^2/2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger]|00\rangle, \tag{23}$$

which is the Fock representation of $|\xi\rangle$. One can check that $|\xi\rangle$ really obeys the eigenvector equations [13]

$$(\hat{Q}_1 + \hat{Q}_2)|\xi\rangle = \sqrt{2}\xi_1|\xi\rangle, \quad (\hat{P}_1 - \hat{P}_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle. \tag{24}$$

This is the concise approach for deriving the entangled state representation, which is based on the generalized Radon transformation of the Wigner operator. Using the IWOP technique, we have the completeness relation

$$\int |\xi\rangle\langle\xi| d^2\xi = 1. \tag{25}$$

Now, we make another Radon transformation of the 3-mode Wigner operator. Since $(\frac{\hat{P}_1}{\mu_1}-\frac{\hat{P}_2}{\mu_2})$, $(\frac{\hat{P}_1}{\mu_1}-\frac{\hat{P}_3}{\mu_3})$ and $(\mu_1\hat{Q}_1+\mu_2\hat{Q}_2+\mu_3\hat{Q}_3)$ are compatible operator, where $\sum_{i=1}^{3}\mu_i=1$ with μ_i being positive real, we can suppose the following eigenvector equations

$$\left(\frac{\hat{P}_1}{\mu_1}-\frac{\hat{P}_2}{\mu_2}\right)|\chi,\rho_1,\rho_2\rangle=\rho_2|\chi,\rho_1,\rho_2\rangle, \tag{26}$$

$$\left(\frac{\hat{P}_1}{\mu_2}-\frac{\hat{P}_3}{\mu_3}\right)|\chi,\rho_1,\rho_2\rangle=\rho_3|\chi,\rho_1,\rho_2\rangle, \tag{27}$$

and

$$(\mu_1\hat{Q}_1+\mu_2\hat{Q}_2+\mu_3\hat{Q}_3)|\chi,\rho_1,\rho_2\rangle=\chi|\chi,\rho_1,\rho_2\rangle, \tag{28}$$

where $|\chi,\rho_1,\rho_2\rangle$ is their common eigenvector, which is to be obtained. Simiarly, according the above theorem, we have

$$\begin{aligned}
&|\chi,\rho_1,\rho_2\rangle\langle\chi,\rho_1,\rho_2| \\
&\quad=\int\prod_{i=1}^{3}dq_i dp_i\delta\left[\rho_1-\left(\frac{\rho_1}{\mu_1}-\frac{\rho_2}{\mu_2}\right)\right]\delta\left[\rho_2-\left(\frac{\rho_1}{\mu_2}-\frac{\rho_3}{\mu_3}\right)\right] \\
&\qquad\times\delta[\chi-(\mu_1 q_1+\mu_2 q_2+\mu_3 q_3)] \\
&\qquad\times\frac{1}{\pi^3}:\ \exp\left[-\sum_{j=1}^{3}[(q_j-\hat{Q}_j)^2+(p_j-\hat{P}_j)^2]\right]: \\
&\quad=\frac{1}{\pi^{3/2}}\frac{\mu_1\mu_2\mu_3}{\lambda}:\ \exp\left[-\frac{1}{\lambda}[\mu_1\mu_2\rho_2-(\mu_2\hat{P}_1-\mu_1\hat{P}_2)]^2\right. \\
&\qquad-\frac{1}{\lambda}[\mu_1\mu_3\rho_3-(\mu_3\hat{P}_1-\mu_1\hat{P}_3)]^2-\frac{1}{\lambda}[\chi-(\mu_1\hat{Q}_1+\mu_2\hat{Q}_2+\mu_3\hat{Q}_3)]^2 \\
&\qquad\left.-\frac{1}{\lambda}[\mu_2\mu_3(\rho_3-\rho_2)-(\mu_3\hat{P}_2-\mu_2\hat{P}_3)]^2\right]:,
\end{aligned} \tag{29}$$

where $\lambda\equiv\mu_1^2+\mu_2^2+\mu_3^2$. Due to (21) and

$$|000\rangle\langle 000|=:\ \exp\left(-\sum_{i=1}^{3}a_i^\dagger a_i\right):, \tag{30}$$

we see that the decomposition of the right-hand side of (29) leads to

$$\begin{aligned}
&|\chi,\rho_1,\rho_2\rangle \\
&\quad=\frac{\sqrt{\frac{\mu_1\mu_2\mu_3}{\lambda}}}{\pi^{3/4}}\exp\left[-\frac{1}{2\lambda}[(\mu_1^2+\mu_3^2)\mu_2^2\rho_2^2+(\mu_1^2+\mu_2^2)\mu_3^2\rho_3^2-2\mu_2^2\mu_3^2\rho_2\rho_3]\right. \\
&\qquad-\frac{\chi^2}{2\lambda}+\frac{\sqrt{2}\chi}{2}\sum_{j=1}^{3}\mu_j a_j^\dagger+\frac{i\sqrt{2}\mu_2\rho_2}{\lambda}[\mu_1\mu_2 a_1^\dagger+\mu_2\mu_3 a_3^\dagger-(\mu_1^2+\mu_3^2)a_2^\dagger]
\end{aligned}$$

$$+\sum_{j=1}^{3}\left(\frac{1}{2}-\frac{\mu_j^2}{\lambda}\right)a_j^{\dagger 2}+\frac{i\sqrt{2}\mu_3\rho_3}{\lambda}[\mu_1\mu_3a_1^{\dagger}+\mu_2\mu_3a_2^{\dagger}-(\mu_1^2+\mu_2^2)a_3^{\dagger}]$$

$$\left.-\frac{2}{\lambda}\sum_{j<k=1}^{3}\mu_j\mu_k a_j^{\dagger}a_k^{\dagger}\right]|000\rangle, \tag{31}$$

which is the same to that of [14]. However, this is a very direct way to find the Fock representation of the tripartite entangled state $|\chi,\rho_1,\rho_2\rangle$. In addition, using the completeness of the Wigner operator, i.e. [8, 9]

$$\int\prod_{i=1}^{n}dp_i dq_i\widehat{\Delta}_n(q_i,p_i)=1 \tag{32}$$

is easily proved that $|\chi,\rho_1,\rho_2\rangle$ has a complete set, i.e.,

$$\begin{aligned}
&\int d\rho_1 d\rho_2 d\chi|\chi,\rho_1,\rho_2\rangle\langle\chi,\rho_1,\rho_2|\\
&\quad=\int d\rho_1 d\rho_2 d\chi\int\prod_{i=1}^{3}dq_i dp_i\delta[\rho_1-(p_1-p_2)]\\
&\qquad\times\delta[\rho_2-(p_1+p_2+p_3)]\delta[\chi-(x_1-2x_2+x_3)]\Delta_3(q_i,p_i)\\
&\quad=\int\prod_{i=1}^{3}dq_i dp_i\Delta_3(q_i,p_i)=1,
\end{aligned} \tag{33}$$

where we directly adopt the properties of the Dirac-detal function.

In summary, by virtue of the Weyl correspondence (Weyl quantization scheme) we have presented the theorem and lemma, which is called the generalized Radon transformation of the Wigner operator. It shows from the examples that the theorem is a new concise approach for obtaining the Fock representation of multi-partite entangled states of continuum variables. This available approach for effectively finding many new quantum mechanical representations may enrich Dirac's representation and transformation theory.

References

1. Deans, S.R.: The Radon Transform and some its Applications. Wiley, New York (1983)
2. Helgason, S.: The Radom Transform. Birkhauser, Boston (1980)
3. Vogel, K., Risken, H.: Phys. Rev. A **40**, 2847 (1989)
4. Smithey, D.T., Beck, M., Raymer, M.G., Faridani, A.: Phys. Rev. Lett. **70**, 1244 (1993)
5. Fan, H.Y., Yu, G.C.: Mod. Phys. Lett. A **15**, 499 (2000)
6. Ellinas, D., Bracken, A.J.: Phys. Rev. A **78**, 052106 (2008)
7. Fan, H.Y., Lv, C.H.: J. Math. Phys. **50**, 102108 (2009)
8. Wigner, E.P.: Phys. Rev. **40**, 74 (1932)
9. Fan, H.Y., Zaidi, H.R.: Phys. Lett. A **124**, 303 (1987)
10. Fan, H.Y., Lu, H.L., Fan, Y.: Ann. Phys. **321**, 480 (2006)
11. Weyl, H.: Z. Phys. **46**, 1 (1927) (1927)
12. Fan, H.Y.: Ann. Phys. **323**, 500 (2008)
13. Fan, H.Y., Klauder, J.R.: Phys. Rev. A **49**, 704 (1994)
14. Fan, H.Y., Jiang, N.Q.: Chin. Phys. Lett. **19**, 1403 (2002)